

EMBEDDING TOPOLOGICAL FRACTALS IN UNIVERSAL SPACES

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ABSTRACT. Let X be a universal (Urysohn) space. We prove that every topological fractal is homeomorphic (isometric) to the attractor $A_{\mathcal{F}}$ of a function system \mathcal{F} on X consisting of Rakotch contractions.

1. INTRODUCTION

Let X be a topological space. By a *function system* on X we shall understand any finite family \mathcal{F} of continuous self-maps of X . Every function system \mathcal{F} generates the mapping

$$\mathcal{F} : \mathbb{K}(X) \rightarrow \mathbb{K}(X), \quad \mathcal{F} : K \mapsto \bigcup_{f \in \mathcal{F}} f(K),$$

on the hyperspace $\mathbb{K}(X)$ of non-empty compact subsets of X , endowed with the Vietoris topology. If the topology of X is generated by a metric d , then the Vietoris topology on $\mathbb{K}(X)$ is generated by the Hausdorff metric

$$d_H(A, B) = \max\{\max_{a \in A} d(a, B), \max_{b \in B} d(A, b)\}.$$

We shall say that a compact set $A \in \mathbb{K}(X)$ is an *attractor* of a function system \mathcal{F} if $\mathcal{F}(A) = A$ and for every compact set $K \in \mathbb{K}(X)$ the sequences of iterations $\mathcal{F}^n(K) = \mathcal{F} \circ \dots \circ \mathcal{F}(K)$ converges to A in the hyperspace $\mathbb{K}(X)$. A function system \mathcal{F} on a Hausdorff topological space X can have at most one attractor, which will be denoted by $A_{\mathcal{F}}$. The following classical result of a Hutchinson-Barnsley theory of fractals [10] detects function systems possessing attractors.

Theorem 1.1. *Each function system \mathcal{F} consisting of Banach contractions of a complete metric space X has a unique attractor $A_{\mathcal{F}}$.*

The proof of Theorem 1.1 given in [10] (cf. also [3]) uses the observation that a Banach contracting function system \mathcal{F} (i.e., a function system consisting of Banach contractions) on a complete metric space X induces a Banach contracting map $\mathcal{F} : \mathbb{K}(X) \rightarrow \mathbb{K}(X)$ of the hyperspace, which makes possible to apply the Banach Contracting Principle to show that \mathcal{F} has an attractor $A_{\mathcal{F}}$.

It turns out that the Banach contractivity of \mathcal{F} in Theorem 1.1 can be weakened to the φ -contractivity, which is defined as follows.

A map $f : X \rightarrow Y$ between metric spaces (X, d_X) and (Y, d_Y) is called φ -contracting for a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ if $d_Y(f(x), f(x')) \leq \varphi(d_X(x, x'))$ for every points $x, x' \in X$. It follows that $f : X \rightarrow Y$ is Banach contracting (i.e., it is a Lipschitz mapping with the Lipschitz constant less than 1) if and only if it is φ -contracting for some function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\sup_{0 < t < \infty} \varphi(t)/t < 1$.

A function $f : X \rightarrow Y$ is called

- *Rakotch contracting* if f is φ -contracting for a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\sup_{a < t < \infty} \varphi(t)/t < 1$ for every $a > 0$;
- *Matkowski contracting* if f is φ -contracting for a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for every $t > 0$, where φ^n denotes the n th iteration of φ ;
- *Edelstein contracting* if $d_Y(f(x), f(x')) < d_X(x, x')$ for any distinct points $x, x' \in X$.

It is known ([11]) that Rakotch contracting maps are Matkowski contracting, Matkowski contracting are Edelstein contracting and that if X is compact, then $f : X \rightarrow Y$ is Rakotch contracting if and only if it is Edelstein contracting. The notions of Rakotch, Matkowski and Edelstein contracting maps are connected with certain generalizations of the Banach Contracting Principle, cf. [8], [14], [19].

2010 *Mathematics Subject Classification.* Primary: 28A80; Secondary: 37C25, 37C70.

Key words and phrases. Topological fractal, universal Urysohn space, Banach contraction, Rakotch contraction.

A topological version of Theorem 1.1 was recently proved by Mihail [16] who introduced the following notion (cf. also [4] for a particular version of it): A function system \mathcal{F} on a Hausdorff topological space X is called *topologically contracting* if

- for every $K \in \mathbb{K}(X)$, there is $D \in \mathbb{K}(X)$ such that $K \subset D$ and $\mathcal{F}(D) \subset D$;
- for every $D \in \mathbb{K}(X)$ with $\mathcal{F}(D) \subset D$, and a sequence $\vec{f} = (f_n)_{n \in \omega} \in \mathcal{F}^\omega$, the set $\bigcap_{n \in \omega} f_0 \circ \dots \circ f_n(D)$ is a singleton.

It can be seen that in this case the singleton $\{\pi(\vec{f})\} = \bigcap_{n \in \omega} f_0 \circ \dots \circ f_n(D)$ does not depend on the choice of the compact set D and the map $\pi : \mathcal{F}^\omega \rightarrow X$, $\pi : \vec{f} \mapsto \pi(\vec{f})$, is continuous (here \mathcal{F}^ω carries the topology of Tychonoff product of countably many copies of the finite space \mathcal{F} endowed with the discrete topology). Moreover, the compact metrizable space $A_{\mathcal{F}} = \pi(\mathcal{F}^\omega)$ is the attractor of the function system \mathcal{F} . This fact was proved by Mihail [16]:

Theorem 1.2. *Every topologically contracting function system \mathcal{F} on a Hausdorff topological space X has an attractor $A_{\mathcal{F}}$, which is a compact metrizable space.*

A Hausdorff topological space X is called a *topological fractal* if $X = \bigcup_{f \in \mathcal{F}} f(X)$ for some topologically contracting function system \mathcal{F} on X . It follows that for every topologically contractive function system \mathcal{F} on a Hausdorff topological space X its attractor $A_{\mathcal{F}}$ is a topological fractal. Mihail's Theorem 1.2 implies that each topological fractal is a compact metrizable space. Moreover, the topology of X is generated by a metric d making all maps $f \in \mathcal{F}$ Rakotch contracting (see [2] or [17]). Topological fractals were introduced and investigated by Kameyama [12] (who called them *self similar sets*) and considered also in [1] and [7].

In this paper we shall search for copies of (Banach) topological fractals in universal (metric) spaces. A topological space X will be called *topologically universal* if every compact metrizable space K admits a topological embedding into X . The following realization theorem will be proved in Section 2.

Theorem 1.3. *If a Tychonoff space X is topologically universal, then every topological fractal is homeomorphic to the attractor $A_{\mathcal{F}}$ of a topologically contractive function system \mathcal{F} on X . If the space X is metrizable, then we can additionally assume that all maps $f \in \mathcal{F}$ are Rakotch contracting with respect to some bounded metric d generating the topology on X .*

For the universal Urysohn space we can prove a bit more. Let us recall that the *universal Urysohn space* is a separable complete metric space \mathbb{U} such that each isometric embedding $f : B \rightarrow \mathbb{U}$ of a subspace B of a finite metric space A extends to an isometric embedding $\bar{f} : A \rightarrow \mathbb{U}$. By [20], a universal Urysohn space exists and is unique up to a bijective isometry.

A compact metric space X will be called a *(Banach) Rakotch fractal* if $X = \bigcup_{f \in \mathcal{F}} f(X)$ for some function system \mathcal{F} consisting of (Banach) Rakotch contractions of X . By [2] and [17], each topological fractal is homeomorphic to a Rakotch fractal. On the other hand, there are examples of Rakotch fractals which are not homeomorphic to Banach fractals (see [1], [12], [18]). The following realization theorem will be proved in Section 3.

Theorem 1.4. *Each (Banach) Rakotch fractal X is isometric to the attractor $A_{\mathcal{F}}$ of a function system \mathcal{F} consisting of Banach (Rakotch) contractions of the universal Urysohn space \mathbb{U} .*

2. COPIES OF TOPOLOGICAL FRACTALS IN UNIVERSAL SPACES

In this section we shall prove Theorem 1.3. At first we need to recall some information on spaces of probability measures.

For a compact metric space (X, d_X) by PX we shall denote the space of Borel probability measures on X endowed with the metric

$$d_{PX}(\mu, \eta) := \inf \left\{ \int_{X \times X} d_X(x, y) d\lambda : \lambda \in \mathcal{B}(\mu, \eta) \right\},$$

where $\mathcal{B}(\mu, \eta)$ is the space of all Borel probability measures on $X \times X$ such that $\pi_1(\lambda) = \mu$ and $\pi_2(\lambda) = \eta$ (here π_1 and π_2 stand for the projections onto the first and the second coordinate, respectively). It is known that (PX, d_{PX}) is a compact metric space and for every measures $\mu, \eta \in PX$, there is $\lambda \in \mathcal{B}(\mu, \eta)$ such that

$d_{PX}(\mu, \eta) = \int_{X \times X} d(x, y) d\lambda$ (cf. [5, Chapter 8, especially Theorem 8.9.3, Theorem 8.10.45, p. 234]; here we use the fact that every Borel probability measure on a compact metrizable spaces is Radon [5, p. 70]).

It follows that the mapping $X \ni x \rightarrow \delta_x \in PX$ assigning to each point $x \in X$ the Dirac measure δ_x supported at x is an isometric embedding of X into PX . Every continuous map $f : X \rightarrow Y$ between compact metric spaces (X, d_X) and (Y, d_Y) induces a continuous map $Pf : PX \rightarrow PY$ between their spaces of probability measures. The map Pf assigns to each measure $\mu \in PX$ the measure $Pf(\mu) \in PY$ defined by $Pf(\mu)(B) = \mu(f^{-1}(B))$ for a Borel subset $B \subset Y$.

Lemma 2.1. *If a map $f : X \rightarrow Y$ between compact metric spaces X, Y is Rakotch contracting, then the induced map $Pf : P(X) \rightarrow P(Y)$ is Rakotch contracting too.*

Proof. Being Rakotch contracting, the map f is φ -contracting for some function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $c_\delta = \sup_{\delta \leq t < \infty} \varphi(t)/t < 1$ for every $\delta > 0$. By the compactness of PX , the Rakotch contractivity of Pf is equivalent to its Edelstein contractivity. So, it suffices to prove that $d_{PY}(Pf(\mu), Pf(\eta)) < d_{PX}(\mu, \eta)$ for any distinct measures $\mu, \eta \in PX$. So take any distinct $\mu, \eta \in PX$. As stated earlier, there is a measure $\lambda \in \mathcal{B}(\mu, \eta)$ such that $d_{PX}(\mu, \eta) = \int_{X \times X} d_X(x, y) d\lambda$. Since $d_{PX}(\mu, \eta) > 0$, for some $\delta > 0$ the compact set $X_\delta = \{(x, x') \in X \times X : d_X(x, x') \geq \delta\}$ has positive measure $\lambda(X_\delta) > 0$.

Let $\tilde{\lambda} \in P(Y \times Y)$ be the image of the measure λ under the map $f \times f : X \times X \rightarrow Y \times Y$, $f \times f : (x, y) \mapsto (f(x), f(y))$. Observe that for every Borel subset $B \subset Y$ we get

$$\tilde{\lambda}(B \times Y) = \lambda((f \times f)^{-1}(B \times Y)) = \lambda(f^{-1}(B) \times f^{-1}(Y)) = \lambda(f^{-1}(B) \times X) = \mu(f^{-1}(B)) = Pf(\mu)(B)$$

and similarly

$$\tilde{\lambda}(Y \times B) = Pf(\eta)(B),$$

which means that $\tilde{\lambda} \in \mathcal{B}(Pf(\mu), Pf(\eta))$. Moreover,

$$d_{PY}(Pf(\mu), Pf(\eta)) \leq \int_{Y \times Y} d_Y(y, y') d\tilde{\lambda} = \int_{X \times X} d_X(f(x), f(x')) d\lambda < \int_{X \times X} d_X(x, x') d\lambda = d_{PX}(\mu, \eta).$$

The last strict inequality follows from $\lambda(X_\delta) > 0$ and the fact that $d_Y(f(x), f(x')) \leq c_\delta \cdot d_X(x, x') < d_X(x, x')$ for every $(x, x') \in X_\delta$. \square

We shall also need a metrization theorem for globally contracting function systems, proved in [2]. A function system \mathcal{F} on a Hausdorff topological space X is called *globally contracting* ([2, Definition 2.1]) if there exists a non-empty compact set $K \subset X$ such that $\mathcal{F}(K) \subset K$ and for every open cover \mathcal{U} of X there is $n \in \mathbb{N}$ such that for every map $f \in \mathcal{F}^n = \{f_1 \circ \dots \circ f_n : f_1, \dots, f_n \in \mathcal{F}\}$ the set $f(X)$ is contained in some set $U \in \mathcal{U}$. The following result was proved in [2, Theorem 6.7].

Theorem 2.2. *A function system \mathcal{F} on a metrizable space X is globally contractive if and only if the topology of X is generated by a bounded metric d making all maps $f \in \mathcal{F}$ Rakotch contractive.*

Now we are able to present:

Proof of Theorem 1.3. Assuming that K is a topological fractal, find a topologically contracting function system \mathcal{F} on K such that $K = \bigcup_{f \in \mathcal{F}} f(K)$. By [2, Theorem 6.8], the topology of K is generated by a metric d_K making all maps $f \in \mathcal{F}$ Rakotch contracting. By Lemma 2.1, the function system $P\mathcal{F} = \{Pf : f \in \mathcal{F}\}$ consists of Rakotch contracting self-maps of the metric space (PK, d_{PK}) .

Given any topologically universal Tychonoff space X , identify the compact metrizable space PK with a (closed) subspace of X . The space PK , being a metrizable compact convex subset of a locally convex space, is an absolute retract in the class of Tychonoff spaces (this follows from [6] and Tietze-Urysohn Theorem [9, 2.1.8]). This implies that each map $Pf : PK \rightarrow PK$, $f \in \mathcal{F}$, can be extended to a continuous map $\overline{Pf} : X \rightarrow PK$. The Rakotch contractivity of $P\mathcal{F}$ and Theorem 2.2 implies that the function system $P\mathcal{F}$ is globally contractive and so is the function system $\overline{P\mathcal{F}} = \{\overline{Pf} : Pf \in P\mathcal{F}\}$ (as $\overline{P\mathcal{F}}(X) \subset PK$).

The global contractivity of $\overline{P\mathcal{F}}$ implies the topological contractivity of $\overline{P\mathcal{F}}$ ([2, Theorem 2.2]). Then the function system $\overline{P\mathcal{F}}$ has a unique attractor, which coincides with K by the uniqueness of the fixed point of the map $\overline{P\mathcal{F}} : \mathbb{K}(X) \rightarrow \mathbb{K}(X)$. Therefore, the topological fractal K is homeomorphic to the attractor of the topologically contracting function system $\overline{P\mathcal{F}}$ on X .

If the space X is metrizable, then by Theorem 2.2, the topology of X is generated by a bounded metric d making all maps $f \in \overline{P\mathcal{F}}$ Rakotch contracting. \square

3. EMBEDDING FRACTALS INTO THE URYSOHN UNIVERSAL SPACE

Recall that the Urysohn space \mathbb{U} is the unique (up to isometry) complete separable metric space \mathbb{U} such that every isometric embedding $f : B \rightarrow \mathbb{U}$ of a finite subset $B \subset \mathbb{U}$ extends to an isometric embedding $\bar{f} : \mathbb{U} \rightarrow \mathbb{U}$, and any separable metric space is isometric to a subspace of \mathbb{U} . According to [15, Theorem 4.1], the universal Urysohn space has a stronger universality property: every isometric embedding $f : B \rightarrow \mathbb{U}$ of a compact subspace $B \subset \mathbb{U}$ extends to an isometric embedding $\bar{f} : \mathbb{U} \rightarrow \mathbb{U}$. In fact, isometric embeddings in this result can be replaced by maps with given oscillation.

For a map $f : X \rightarrow Y$ between two metric spaces (X, d_X) and (Y, d_Y) its *oscillation* is the function $\omega_f : [0, \infty] \rightarrow [0, \infty]$ assigning to each $\delta \in [0, \infty]$ the number

$$\omega_f(\delta) = \sup\{d_Y(f(x), f(x')) : x, x' \in X, d_X(x, x') \leq \delta\} \in [0, \infty].$$

It follows that $d_Y(f(x), f(x')) \leq \omega_f(d_X(x, x'))$ for every points $x, x' \in X$. It is clear that the map $f : X \rightarrow Y$ is uniformly continuous if and only if $\lim_{\delta \rightarrow 0} \omega_f(\delta) = 0$.

If the metric space (X, d_X) is *geodesic* (in the sense that for every points $x, x' \in X$ there is an isometric embedding $\gamma : [0, d_X(x, x')] \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(d_X(x, x')) = x'$), then for any map $f : X \rightarrow Y$ its oscillation ω_f is *subadditive* in the sense that $\omega_f(s + t) \leq \omega_f(s) + \omega_f(t)$ for any $s, t \in [0, \infty]$. This motivates the following definition.

By a *continuity modulus* we shall understand any continuous subadditive function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$. It is easy to see that each continuous, concave function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$ is a continuity modulus. The following lemma uses some ideas from [13].

Lemma 3.1. *Let φ be a continuity modulus. Every (injective) map $f : B \rightarrow \mathbb{U}$ with $\omega_f \leq \varphi$ defined on a compact subset B of a separable metric space (A, d_A) extends to an (injective) map $\bar{f} : A \rightarrow \mathbb{U}$ with continuity modulus $\omega_{\bar{f}} \leq \varphi$.*

Proof. Fix a countable dense subset $\{a_n\}_{n \in \mathbb{N}}$ in A and put $B_0 = B$ and $B_{n+1} = B_n \cup \{a_{n+1}\}$ for $n \in \omega$. Let $f_0 = f$. By induction we shall construct a sequence of (injective) maps $(f_n : B_n \rightarrow \mathbb{U})_{n \in \omega}$ such that $f_{n+1}|_{B_n} = f_n$ and $\omega_{f_n} \leq \varphi$ for all $n \in \omega$. Assume that for some $n \in \omega$ the map $f_n : B_n \rightarrow \mathbb{U}$ has been constructed. Consider the compact space $B_{n+1} = B_n \cup \{a_{n+1}\}$. If $B_{n+1} = B_n$, then put $f_{n+1} = f_n$. If $B_{n+1} \neq B_n$, then $a_{n+1} \notin B_n$. Consider the subspace $f_n(B_n) \subset \mathbb{U}$ of the Urysohn space and fix any point $y \notin f_n(B_n)$. On the union $Y = f_n(B_n) \cup \{y\}$ consider the metric d_Y which coincides on $f_n(B_n)$ with the metric $d_{\mathbb{U}}$ of the Urysohn space \mathbb{U} and

$$d_Y(z, y) = \min\{d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b, a_{n+1})) : b \in B_n\}$$

for $z \in f_n(B_n)$. It follows from the compactness of B_n and $a_{n+1} \notin B_n$ that $d_Y(z, y) > 0$ for every $z \in f_n(B_n)$. Let us show that the metric d_Y satisfies the triangle inequality and hence is well-defined. Indeed, for any points $z, z' \in f_n(B_n)$, we can find points $b, b' \in B_n$ such that $d_Y(z, y) = d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b, a_{n+1}))$ and $d_Y(z', y) = d_{\mathbb{U}}(z', f_n(b')) + \varphi(d_A(b', a_{n+1}))$. Then

$$\begin{aligned} d_Y(z, z') &= d_{\mathbb{U}}(z, z') \leq d_{\mathbb{U}}(z, f_n(b)) + d_{\mathbb{U}}(f_n(b), f_n(b')) + d_{\mathbb{U}}(f_n(b'), z') \leq \\ &\leq d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b, b')) + d_{\mathbb{U}}(f_n(b'), z') \leq \\ &\leq d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b, a_{n+1}) + d_A(a_{n+1}, b')) + d_{\mathbb{U}}(f_n(b'), z') \leq \\ &\leq d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b, a_{n+1})) + \varphi(d_A(a_{n+1}, b')) + d_{\mathbb{U}}(f_n(b'), z') = d_Y(z, y) + d_Y(y, z'). \end{aligned}$$

On the other hand,

$$d_Y(z, y) \leq d_{\mathbb{U}}(z, f_n(b')) + \varphi(d_A(b', a_{n+1})) \leq d_{\mathbb{U}}(z, z') + d_{\mathbb{U}}(z', f_n(b')) + \varphi(d_A(b', a_{n+1})) = d_Y(z, z') + d_Y(z', y).$$

So, the metric d_Y satisfies the triangle inequality and hence is well-defined.

By [15, Theorem 4.1], the identity embedding $f_n(B_n) \rightarrow \mathbb{U}$ extends to an isometric embedding $e : Y \rightarrow \mathbb{U}$. Let $f_{n+1} : B_{n+1} \rightarrow \mathbb{U}$ be the map such that $f_{n+1}|_{B_n} = f_n$ and $f_{n+1}(a_{n+1}) = e(y)$. Since $e(y) \notin f_n(B_n)$, the map f_{n+1} is injective if so is the map f_n . It follows from $\omega_{f_n} \leq \varphi$ and

$$d_{\mathbb{U}}(f_{n+1}(x), f_{n+1}(a_{n+1})) = d_Y(f_n(x), y) \leq \varphi(d_A(x, a_{n+1})) \quad \text{for } x \in B_n$$

that $\omega_{f_{n+1}} \leq \varphi$. This completes the inductive step.

After the completion of the inductive construction, consider the map $f_\omega : B_\omega \rightarrow \mathbb{U}$ defined on the set $B_\omega = \bigcup_{n \in \omega} B_n$ by $f_\omega|_{B_n} = f_n$ for $n \in \omega$. It follows from $\omega_{f_n} \leq \varphi$, $n \in \omega$, that $\omega_{f_\omega} \leq \varphi$. This implies that the map f_ω is uniformly continuous and hence extends to a uniformly continuous map $\bar{f} : A \rightarrow \mathbb{U}$ having the oscillation $\omega_{\bar{f}} \leq \varphi$. It is clear that $\bar{f}|_B = f_0 = f$. \square

Now we are able to present the *Proof of Theorem 1.4*.

Given a (Banach) Rakotch fractal X , choose a function system \mathcal{F} consisting of (Banach) Rakotch contractions of X such that $X = \bigcup_{f \in \mathcal{F}} f(X)$. It follows that all maps $f \in \mathcal{F}$ are φ -contracting for some continuity modulus φ such that $\sup_{t \geq \delta} \varphi(t)/t < 1$ for all $\delta > 0$. (Moreover, if all maps $f \in \mathcal{F}$ are Banach contractions, then we can assume that $\sup_{t > 0} \varphi(t)/t < 1$).

Since the universal Urysohn space \mathbb{U} contains an isometric copy of each compact metric space, we can assume that X is a subspace of \mathbb{U} . By Lemma 3.1, each map $f \in \mathcal{F}$ extends to a map $\bar{f} : \mathbb{U} \rightarrow \mathbb{U}$ such that $\omega_{\bar{f}} \leq \varphi$, which implies that \bar{f} is a (Banach) Rakotch contraction of \mathbb{U} . By Theorems 1.2 and 2.2, the function system $\bar{\mathcal{F}} = \{\bar{f} : f \in \mathcal{F}\}$ on the universal Urysohn space \mathbb{U} has an attractor $A_{\bar{\mathcal{F}}}$, which is a unique fixed point of the map $\bar{\mathcal{F}} : \mathbb{K}(\mathbb{U}) \rightarrow \mathbb{K}(\mathbb{U})$ on the hyperspace $\mathbb{K}(\mathbb{U})$ of the Urysohn space \mathbb{U} . Taking into account that $\bar{\mathcal{F}}(X) = \bigcup_{f \in \mathcal{F}} \bar{f}(X) = \bigcup_{f \in \mathcal{F}} f(X) = X$, we conclude that $X = A_{\bar{\mathcal{F}}}$.

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